

A Classification of Orientable Regular Embeddings of Complete Multipartite Graphs

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Abstract

Let $K_{m[n]}$ be the complete multipartite graph with m parts, while each part contains n vertices. The regular embeddings of complete graphs $K_{m[1]}$ (see [1, 13, 23]) and complete bipartite graphs $K_{2[n]}$ (see [5, 6, 14, 15, 16, 17, 18, 21, 19]) have been determined. Based on our former paper [24], this paper gives a complete classification of orientable regular embeddings of graphs $K_{m[n]}$ where $m \geq 3$ and $n \geq 2$.

1 Introduction

A (topological) *map* is a cellular decomposition of a closed surface. A common way to describe such a map is to view it as a 2-cell embedding of a connected graph or multigraph Γ into the surface S . The components of the complement $S \setminus \Gamma$ are simply-connected regions called the *faces* of the map (or the embedding). An *automorphism* of a map \mathcal{M} is an automorphism of the underlying (multi)graph Γ which extends to a self-homeomorphism of the supporting surface S . It is well known that the automorphism group $\text{Aut}(\mathcal{M})$ of a map \mathcal{M} acts semi-regularly on the set of all incident vertex-edge-face triples (or *flags* of Γ). In particular, if $\text{Aut}(\mathcal{M})$ acts regularly on the flags, we call it a *regular map*. In the orientable case, if the group of all orientation-preserving automorphisms of \mathcal{M} acts regularly on the set of all incident vertex-edge pairs (or *arcs*) of \mathcal{M} , then we call \mathcal{M} an *orientable regular map*. Such maps fall into two classes: those that admit also orientation-reversing automorphisms, which are called *reflexible*, and those that do not, which are *chiral*. Therefore, a reflexible map is a regular map but a chiral map is not.

One of the central problems in topological graph theory is to classify all the regular embeddings in orientable or nonorientable surfaces of a given graph. In a general setting, the classification problem was treated by Gardiner, Nedela, Širáň and Škoviera in [9]. However, for particular classes of graphs, it has been solved only in a few cases. Let $K_{m[n]}$ be the complete multipartite graph with m parts, while each part contains n vertices.

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All the regular embeddings of complete graphs $K_{m[1]}$ have been determined by Biggs, James and Jones [1, 13] for orientable case and by Wilson [23] for nonorientable case. As for the complete bipartite graphs $K_{2[n]}$, the nonorientable regular embeddings of these graphs have recently been classified by Kwak and Kwon [19]; during the past twenty years, several papers [5, 6, 15, 16, 17, 18, 21] contributed to the orientable case, and the final classification was given by Jones [14] in 2010. Since then, the classification for general case $m \geq 3$ and $n \geq 2$ has become an attractive topic in this research field. The only known result is the determination of orientable regular embeddings of the graphs $K_{m[p]}$ (where p is a prime) given by Du, Kwak and Nedela in [8].

In this paper, we shall focus on the orientable regular embeddings of complete multipartite graphs. A start point is the main theorem in our former paper [24], namely the reduction theorem as follows:

Proposition 1.1. *Let \mathcal{M} be a orientable regular embedding of $K_{m[n]}$ where $m \geq 3$ and $n \geq 2$, and let $\text{Aut}_0(\mathcal{M})$ be the subgroup of $\text{Aut}(\mathcal{M})$ fixing each part setwise. Then $\text{Aut}_0(\mathcal{M})$ is an isobicyclic group. Moreover, we have*

- (1) *if $m \geq 4$, then $m = p$ and $n = p^e$ for some prime p ; or*
- (2) *if $m = 3$, then $\text{Aut}_0(\mathcal{M}) = Q \times K$, where Q is a 3-subgroup (may be identify group) and K is an abelian 3'-subgroup.*

In Proposition 1.1, an isobicyclic group means a group $H = \langle x \rangle \langle y \rangle$, where $|x| = |y| = n$, $\langle x \rangle \cap \langle y \rangle = 1$ and $x^\alpha = y$ for an involution $\alpha \in \text{Aut}(H)$. Throughout the paper, we call (H, x, y) a n -isobicyclic triple and it plays an important role in the classification of regular embeddings of $K_{m[n]}$.

As usual, the orientable regular map will be presented by a triple $(G; a, b)$, where $G = \langle a, b \rangle$ and $|b| = 2$, for the details see Section 2. Now we are ready to state the main theorem of this paper.

Theorem 1.2. *For $m \geq 3$ and $n \geq 2$, let $K_{m[n]}$ be the complete multipartite graph with m parts, while each part contains n vertices. Let \mathcal{M} be an orientable regular embedding of $K_{m[n]}$. Let $G = \text{Aut}(\mathcal{M})$, $H = \text{Aut}_0(\mathcal{M})$, the subgroup of $\text{Aut}(\mathcal{M})$ fixing each part setwise and P a Sylow p -subgroup of G . Then G and \mathcal{M} are given by*

- (1) *$m = p \geq 3$, $n = p^e$ for a prime p and an integer $e \geq 1$, $H \cong \mathbb{Z}_p^2$ for $e = 1$ and H is nonabelian for $e \geq 2$, $\text{Exp}(P) = p^{e+1}$:*

$$G_1 = G_1(p, e) = \langle a, c | a^{p^e(p-1)} = c^{p^{e+1}} = 1, c^a = c^r \rangle, \text{ where } \mathbb{Z}_{p^{e+1}}^* = \langle r \rangle;$$

$$\mathcal{M}_1(p, e, j) = \mathcal{M}(G_1; a^j, a^{\frac{p^e(p-1)}{2}}c), \text{ where } j \in \mathbb{Z}_{p^e(p-1)}^*;$$
- (2) *$m = n = p \geq 3$, for a prime p , $H \cong \mathbb{Z}_p^2$ and $\text{Exp}(P) = p$:*

$$G_2 = G_2(p) = \langle w, z, c, g \mid \langle w, z \rangle \cong \mathbb{Z}_p^2, c^p = g^{p-1} = 1, c^g = c^t, w^c = wz, z^c = z, w^g = w, z^g = z^t \rangle, \text{ where } \mathbb{Z}_p^* = \langle t \rangle;$$

$$\mathcal{M}_2(p, j) = \mathcal{M}(G_2; (wg)^j, (wg)^{\frac{p-1}{2}}c), \text{ where } j \in \mathbb{Z}_{p-1}^*;$$

- (3) $m = p = 3, n = k3^e$ for $3 \nmid k$, either $e = 0, 1$ and $k \geq 2$ or $e \geq 2$, and H is abelian:

$$G_3 = G_3(k, e, l) = \langle a, b \mid a^{2 \cdot 3^e k} = b^2 = 1, a^2 = x, x^b = y, [x, y] = 1, y^a = x^{-1}y^{-1}, (ab)^3 = x^{3^{e-1}kl}y^{-3^{e-1}kl} \rangle, \text{ where } l = 0 \text{ for } e = 0; \text{ and } l = 0, 1 \text{ for } e \geq 1;$$

$$\mathcal{M}_3(k, e, l, j) = \mathcal{M}(G_3; a^j, b), \text{ where } (l, j) = (0, 1), (1, \pm 1);$$

- (4) $m = p = 3, n = k3^e$ for $3 \nmid k, k \geq 2, e \geq 2, H$ is nonabelian, $\text{Exp}(P) = 3^{e+1}$:

$$G_4 = G_4(k, e) = \langle a, b \mid a^{2 \cdot 3^e k} = b^2 = 1, c = a^{3^e}b, a^{2 \cdot 3^e} = x_1, x_1^b = y_1, [x_1, y_1] = 1, y_1^a = x_1^{-1}y_1^{-1}, c^{3^{e+1}} = 1, c^a = c^2x_1^u y_1^{\frac{u-1}{2}} \rangle, \text{ where } u3^e \equiv 1 \pmod{k};$$

$$\mathcal{M}_4(k, e, j) = \mathcal{M}(G_4; a^j, b), \text{ where } j \in \mathbb{Z}_{2k \cdot 3^e}^* \text{ and } \mathcal{M}_4(k, e, j_1) \cong \mathcal{M}_4(k, e, j_2) \text{ if and only if } j_1 \equiv j_2 \pmod{2 \cdot 3^e};$$

- (5) $m = p = 3, n = 9k$ for $3 \nmid k, H$ is nonabelian, $\text{Exp}(P) = 9$:

$$G_5 = G_5(k, l) = \langle a, b \mid a^{18k} = b^2 = 1, a^2 = x, x^b = y, [x, y] = x^{3k}y^{-3k}, y^a = x^{-1}y^{-1}, (ab)^3 = x^{3lk}y^{-3lk} \rangle, \text{ where } l = 0 \text{ or } \pm 1;$$

$$\mathcal{M}_5(k, l, j) = \mathcal{M}(G_5; a^j, b), \text{ where } j = \pm 1.$$

The above maps are unique determined by the given parameters. Table 1 and 2 give the enumerations for these maps.

Table 1:
Enumerations of the resulting maps

Maps	Number	Reflexible or Chiral	Type $\{s, t\}$ s -gon, valency t
$\mathcal{M}_1(p, e, j)$	$p^{e-1}(p-1)\phi(p-1)$	C	$\{p^e(p-1), p^e(p-1)\}$ if $p \equiv 1 \pmod{4}$, $\{\frac{p^e(p-1)}{2}, p^e(p-1)\}$ if $p \equiv 3 \pmod{4}$
$\mathcal{M}_2(p, j)$	$\phi(p-1)$	C	$\{p(p-1), p(p-1)\}$ if $p \equiv 1 \pmod{4}$, $\{\frac{p(p-1)}{2}, p(p-1)\}$ if $p \equiv 3 \pmod{4}$
$\mathcal{M}_3(k, e, 0, 1)$	1	R	$\{3, 2 \cdot 3^e k\}$
$\mathcal{M}_3(k, e, 1, \pm 1)$	2	C	$\{9, 2 \cdot 3^e k\}$
$\mathcal{M}_4(k, e, j)$	$2 \cdot 3^{e-1}$	C	$\{3^{e+1}, 2 \cdot 3^e k\}$
$\mathcal{M}_5(k, l, j)$	6	C	$\{3, 18k\}$ if $l = 0$, $\{9, 18k\}$ if $l = \pm 1$

Table 2:

Total numbers of regular embeddings of $K_{m[n]}$, $n = 3^e k, 3 \nmid k$				
m	n	Reflexible	Chiral	Total
3	k	1	0	1
	$3k$	1	2	3
	$9k$	1	14	15
	$3^e k (e \geq 3)$	1	$2 \cdot 3^{e-1} + 2$	$2 \cdot 3^{e-1} + 3$
$p \geq 5$	p	0	$p\phi(p-1)$	$p\phi(p-1)$
	$p^e (e \geq 2)$	0	$p^{e-1}(p-1)\phi(p-1)$	$p^{e-1}(p-1)\phi(p-1)$

Notion: Throughout this paper, p is an odd prime. The notion $p^d || n$ indicates that p^d is the highest power of p dividing an integer n . We use (s, t) to denote the great common divisor of two integers s and t and $[n]$ to denote the set $\{1, \dots, n\}$ for any positive integer n . By $A : B$ we mean a semi direct product of group A by group B . For a ring S , we use S^* to denote the multiplication group of S . For other notion not defined here, please refer [11, 4].

In Section 2, we shall introduce the orientable regular embeddings of a graph in more details and give some preliminary results. To classify the orientable regular embeddings of $K_{m[n]}$ for all $m \geq 3$ and $n \geq 2$, by Proposition 1.1 we only need to deal with the cases $K_{p[p^e]}$ where p is an odd prime and $K_{3[n]}$ where $n = k3^e$ for $3 \nmid k$ and $k \geq 2$, separately. Therefore, we shall deal with these two cases in Sections 3 and 4 respectively, and prove Theorem 1.2 in Section 5.

2 Preliminaries

Let $\Gamma = \Gamma(V, D)$ be a graph with vertex set $V = V(\Gamma)$ and arc set $D = D(\Gamma)$. By S_V and S_D we denote the symmetric groups on the vertex set and on the arc set, respectively. The involution L in S_D interchanging the two arcs underlying every given edge is called the *arc-reversing involution*. An element R in S_D which cyclically permutes the arcs initiated at v for each vertex $v \in V(\Gamma)$ is called a *rotation*. In the investigation of maps, it is often useful to replace topological maps on orientable surfaces with their combinatorial counterparts. It is well-known that graph embeddings into orientable surfaces can be described by means of rotations. A map \mathcal{M} with underlying graph Γ can be identified with a triple $\mathcal{M} = \mathcal{M}(\Gamma; R, L)$, where R is a rotation and L is the arc-reversing involution of Γ . By the connectivity of Γ , $\text{Mon}(\mathcal{M}) := \langle R, L \rangle$ is a transitive subgroup of S_D . Given two maps $\mathcal{M}_1 = \mathcal{M}(\Gamma_1; R_1, L_1)$ and $\mathcal{M}_2 = \mathcal{M}(\Gamma_2; R_2, L_2)$, a graph isomorphism $\phi : \Gamma_1 \rightarrow \Gamma_2$ is called a *map orientation-preserving isomorphism* from \mathcal{M}_1 to \mathcal{M}_2 if $R_1\phi = \phi R_2$, noting that $L_1\phi = \phi L_2$ holds in any case. In particular, if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, then ϕ is called an *automorphism* of \mathcal{M} . The automorphisms of \mathcal{M} form a group $\text{Aut}(\mathcal{M}) \leq \text{Aut}(\Gamma)$, called the *automorphism group* of the map \mathcal{M} . By this definition, $\text{Aut}(\mathcal{M}) \leq C_{S_D}(\text{Mon}(\mathcal{M}))$, the centralizer of $\text{Mon}(\mathcal{M})$ in S_D . Also $\text{Aut}(\mathcal{M})$ acts semi-regularly on D , which follows

from the transitivity of $\text{Mon}(\mathcal{M})$ on D . If the action is regular, the map \mathcal{M} is called *regular*. As a consequence of some well-known results in a permutation group theory (see [11, I.6.5]), we infer that in a regular map \mathcal{M} , the two associated permutation groups $\text{Aut}(\mathcal{M})$ and $\text{Mon}(\mathcal{M})$ on D can be viewed as the right and the left regular representations of an abstract group G , so that $G \cong \text{Aut}(\mathcal{M}) \cong \text{Mon}(\mathcal{M})$, mutually centralizing each other in S_D .

Suppose Γ is a simple graph. Then we describe a regular map in terms of its automorphism group, and its underlying graph as a coset graph. Let G be a finite group and H a proper subgroup of G with $\cap_{g \in G} H^g = 1$. Let B be a double coset of H in G such that $B = B^{-1}$. From now on, we use $\Gamma = \Gamma(G; H, B)$ to denote the *coset graph* with $V(\Gamma) = \{Hg \mid g \in G\}$ and $D(\Gamma) = \{(Hg, Hbg) \mid b \in B, g \in G\}$. Note that G acts faithfully and arc-transitively on the coset graph by right multiplication. In what follows, the group G is often identified with the corresponding group of right multiplications.

Definition 2.1. Let $G = \langle r, \ell \rangle$ be a finite two-generator group with $\ell^2 = 1$ and $\langle r \rangle \cap \langle r \rangle^\ell = 1$. By an algebraic map $\mathcal{M}(G; r, \ell) = (\Gamma; R)$, we mean the map whose underlying graph is the coset graph $\Gamma = \Gamma(G; \langle r \rangle, \langle r \rangle^\ell \langle r \rangle)$ and rotation R is determined by $e^R = e^{g_1^{-1} r g_1} = (\langle r \rangle g_1, \langle r \rangle g_2 g_1^{-1} r g_1)$ for any arc $e = (\langle r \rangle g_1, \langle r \rangle g_2)$ in $D(\Gamma)$.

By the definition given above, G acts arc-regularly on Γ (by right multiplication) and preserves the rotation R of the map $\mathcal{M}(G; r, \ell)$. Therefore any algebraic map \mathcal{M} is regular, with $\text{Aut}(\mathcal{M}) \cong \text{Mon}(\mathcal{M}) \cong G$. It is a matter of routine to check that every regular embedding of a graph can be described by an algebraic map, and that two such algebraic maps $\mathcal{M}(G; r_1, \ell_1)$ and $\mathcal{M}(G; r_2, \ell_2)$ are isomorphic if and only if there exists an automorphism $\sigma \in \text{Aut}(G)$ such that $r_1^\sigma = r_2$ and $\ell_1^\sigma = \ell_2$.

Finally we give two results used later.

Lemma 2.2. Let $q = 1 + p^f$ where p is a prime and $f \geq 1$. If $p^d \mid k$ where $d \geq 1$, then

- (1) $(q^k - 1)/(q - 1) \equiv k \pmod{p^{d+f}};$
- (2) $(q^{k+1} - 1)/(q - 1) \equiv k + 1 \pmod{p^{d+f}}.$

Proof. (1) Since $(q^k - 1)/(q - 1) = k + \sum_{i=2}^{k-1} \binom{k}{i} p^{(i-1)f}$, it suffices to prove that $p^{d+f} \mid \binom{k}{i} p^{(i-1)f}$ for any $2 \leq i \leq k$.

The conclusion is clear for $i - 1 \geq \frac{d+f}{f}$ and so we assume that $i - 1 < \frac{d+f}{f}$. Then $2 \leq i < 2 + \frac{d}{f} \leq p^d$. Set $k = k' p^d$ and $i = i' p^{d_i}$ where $(p, i') = 1$. Then $0 \leq d_i < d$ and

$$\binom{k}{i} = \binom{k-1}{i-1} \frac{k}{i} = \binom{k-1}{i-1} \frac{k'}{i'} p^{d-d_i}.$$

Since $\binom{k}{i}$ is an integer and $(i', p) = 1$, we have $i' \mid \binom{k-1}{i-1} k'$ and hence $\binom{k-1}{i-1} \frac{k'}{i'}$ is an integer as well. Noting that

$$\binom{k}{i} p^{(i-1)f} = \binom{k-1}{i-1} \frac{k'}{i'} p^{(i-1)f + d - d_i},$$

we have $p^{(i-1)f+d-d_i} \mid \binom{k}{i} p^{(i-1)f}$. Noting $i'p^{d_i} = i \geq 2$, one may check that $i'p^{d_i} - d_i - 1 \geq 1$. Then

$$(i-1)f + d - d_i = (i'p^{d_i} - 1)f + d - d_i \geq (i'p^{d_i} - d_i - 1)f + d \geq f + d,$$

which implies $p^{d+f} \mid \binom{k}{i} p^{(i-1)f}$.

(2) First note by [15, Lemma 6] that if $p^h \parallel (r-1)$, then $p^{h+i} \parallel r^{p^i} - 1$ for any nonnegative integer i . Actually, from Binomial Theorem one may see that $p^{h+i} \parallel r^{mp^i} - 1$ for any m such that $(p, m) = 1$.

Using the above argument, we have $q^k = (1 + p^f)^k \equiv 1 \pmod{p^{d+f}}$. By (1) we get $(q^k - 1)/(q - 1) \equiv k \pmod{p^{d+f}}$. Hence,

$$(q^{k+1} - 1)/(q - 1) = (q^k - 1)/(q - 1) + q^k \equiv (k + 1) \pmod{p^{d+f}}.$$

□

Lemma 2.3. *Let m be an odd prime and $n > 2$ an integer. Let $G = \langle a, b \rangle$ and $H = \langle x, y \rangle$ where $a^{m-1} = x$, $x^b = y$. Suppose that (H, x, y) is a n -isobicyclic triple, $H \trianglelefteq G$, $G/H \cong \text{AGL}(1, m)$ and $C_G(H) = Z(H)$. Then $\mathcal{M}(G; a, b)$ is a regular embedding of $K_{m[n]}$.*

Proof. It suffices to show that the coset graph $\Gamma = \text{Cos}(G, \langle a \rangle, b)$ is a complete m -partite graph. Since (H, x, y) is a n -isobicyclic triple, we have $H = \langle x \rangle \langle y \rangle$, $\langle x \rangle \cap \langle y \rangle = 1$ and $|H| = n^2$. Noting that $|G| = |G/H||H| = |\text{AGL}(1, m)||H| = m(m-1)n^2$, we have $|G : \langle a \rangle| = mn$. Since $a^{m-1} = x$, $(a^b)^{m-1} = (a^{m-1})^b = y$, $H = \langle x, y \rangle$ and $C_G(H) = Z(H)$, we have $\langle a \rangle \cap \langle a^b \rangle \leq C_G(x) \cap C_G(y) = C_G(H) = Z(H) \leq H$. It follows that $\langle a \rangle \cap \langle a^b \rangle \leq H \cap \langle a \rangle \cap \langle a^b \rangle = \langle x \rangle \cap \langle y \rangle = 1$, that is $\langle a \rangle \cap \langle a^b \rangle = 1$. Therefore Γ is a simple graph of order mn and valency $(m-1)n$.

Set $\Delta = \{\langle a \rangle h \mid h \in H\}$ and $\Sigma = \{\Delta g \mid g \in G\}$. Then Σ is a block system for G acting on $V(\Gamma)$. Since $|\langle a \rangle H : \langle a \rangle| = |\langle a \rangle \langle x \rangle \langle y \rangle : \langle a \rangle| = |\langle a \rangle \langle y \rangle : \langle a \rangle| = n$, we have $|\Delta| = n$ and then $|\Sigma| = m$. Clearly, $y^{i-j} \notin b\langle a \rangle$ for any two integers i and j in $[n]$. Noting that $\Delta = \{\langle a \rangle h \mid h \in H\} = \{\langle a \rangle y^i \mid i \in [n]\}$, Δ contains no pair of adjacent vertices. Therefore Γ is a complete m -partite graph. □

3 Regular embeddings of $K_{p[p^e]}$

In this section, we assume that $m = p$ and $n = p^e$ where p is an odd prime and $e \geq 1$. Set $\Gamma = K_{p[p^e]}$, with the vertex set

$$V(\Gamma) = \bigcup_{i=1}^p \Delta_i, \text{ where } \Delta_i = \{\gamma_{i1}, \gamma_{i2}, \dots, \gamma_{ip^e}\}$$

and the edges are all pairs $\{\gamma_{ij}, \gamma_{kl}\}$ of vertices with $i \neq k$. Then $\text{Aut}(\Gamma) = S_{p^e} \wr S_p$, which has blocks Δ_i where $1 \leq i \leq p$.

Let \mathcal{M} be an orientable regular map with the underlying graph Γ . Set $G = \text{Aut}(\mathcal{M}) = \langle a, b \rangle$, where $\langle a \rangle = G_{\gamma_{11}}$ and b reverses the arc $(\gamma_{11}, \gamma_{21})$. Set $H = \text{Aut}_0(\mathcal{M})$, the subgroup of G preserving each part setwise. Then by Proposition 1.1, H is an p^e -isobicyclic group.

By a result of Hupert [10], H is metacyclic. One can see from [15] that

$$H = \langle x, z | x^{p^e} = z^{p^e} = 1, z^x = z^q \rangle, \quad (1)$$

where $q = 1 + p^f$ for $f \in [e]$ and different f give nonisomorphic groups. Moreover, each element of H can be written uniquely in the form $x^i z^j$ where $i, j \in \mathbb{Z}_{p^e}$, with the rule

$$(x^i z^j)(x^k z^l) = x^{i+k} z^{jq^k+l}, \quad (x^i z^j)^k = x^{ik} z^{j(q^{ik}-1)/(q^i-1)} \quad (f \neq e). \quad (2)$$

The center, derived subgroup and Frattini subgroup of H are $Z(H) = \langle x^{p^{e-f}}, z^{p^{e-f}} \rangle$, $H' = \langle z^f \rangle$ and $\Phi(H) = \langle h^p \mid h \in H \rangle$, respectively. The exponent $\text{Exp}(H)$ of H is p^e . Moreover, H is a regular p -group, that is all elements $h_1, h_2 \in H$ satisfy $(h_1 h_2)^p = h_1^p h_2^p c_1^p \cdots c_k^p$ where $c_1, \dots, c_k \in \langle h_1, h_2 \rangle'$.

Since $a^{m-1} \in H$, set $a^{m-1} = z^j x^i$, where $(i, p) = 1$. Then $a^{i^{-1}(m-1)} = z^{j'} x$ for some j' , and $z^{a^{i^{-1}(m-1)}} = z^x = z^{1+q}$. Replacing a and x by $a^{i^{-1}}$ and $z^{j'} x$, respectively, we may always assume that $a^{m-1} = x$. Also set $y = x^b$.

Write $\overline{G} = G/H$ and we use $\overline{\Gamma}$ to denote the quotient (block) graph of Γ induced by H . Clearly, $\overline{\Gamma} \cong K_p$. Let P be a Sylow p -subgroup of G . Then P is an extension of H by \mathbb{Z}_p . Since $\text{Exp}(H) = p^e$, we get $\text{Exp}(P) = p^{e+1}$ or p^e . We shall divide the proof into three subcases, that is $\text{Exp}(P) = p^{e+1}$, $\text{Exp}(P) = p^e$ and H is non abelian, and $\text{Exp}(P) = p^e$ and H is abelian, in the following three subsections.

3.1 $\text{Exp}(P) = p^{e+1}$

Lemma 3.1. *Suppose that $\text{Exp}(P) = p^{e+1}$. Then \mathcal{M} is isomorphic to $\mathcal{M}_1(p, e, j)$ for $p \geq 3$ and $e \geq 1$, where $j \in \mathbb{Z}_{p^e(p-1)}^*$. The number of the maps is $p^{e-1}(p-1)\phi(p-1)$. Moreover, each such map is chiral, with the type $\{p^e(p-1), p^e(p-1)\}$ if $p \equiv 1 \pmod{4}$ and $\{\frac{p^e(p-1)}{2}, p^e(p-1)\}$ if $p \equiv 3 \pmod{4}$.*

Proof. Suppose that $\text{Exp}(P) = p^{e+1}$. The proof is divided into two steps.

(1) Show $G \cong G_1(p, e)$.

Recall that for $p \geq 3$,

$$G_1 = G_1(p, e) = \langle a, c | a^{p^e(p-1)} = c^{p^{e+1}} = 1, c^a = c^r \rangle,$$

where r is a given generator of $\mathbb{Z}_{p^{e+1}}^*$.

Since $\text{Exp}(P) = p^{e+1}$, there exists an element g of order p^{e+1} in $G \setminus H$ permuting p parts. Since the subgroup $\langle g^p \rangle$ of H is abelian and of order p^e , it is regular on each part. Then $\langle g \rangle$ is regular on $V(\Gamma)$. Recalling that $G_\gamma = \langle a \rangle$, we get $\langle a \rangle \cap \langle g \rangle = 1$. Then

$$p^{2e+1}(p-1) = |G| \geq |\langle a \rangle \langle g \rangle| = |\langle a \rangle| |\langle g \rangle| = p^{2e+1}(p-1),$$

which tells $G = \langle a \rangle \langle g \rangle$, a product of two cyclic groups. Then G' is abelian by a theorem in [12]. Set $\overline{G} = G/H$ and $c = a^{\frac{p^e(p-1)}{2}}b$. Then $\overline{G} \cong \mathbb{Z}_p : \mathbb{Z}_{p-1}$. First, we show a fact.

Fact: $G' = \langle c \rangle$ and $G = \langle c \rangle : \langle a \rangle$.

Since $\overline{c} = \overline{a}^{\frac{p^e(p-1)}{2}}\overline{b}$, a product of two involutions, we get $|\langle \overline{c} \rangle| = p$, in viewing of the structure of $\mathbb{Z}_p : \mathbb{Z}_{p-1}$. Thus, c is a p -element in G . In particular, c permutes the p -parts. This implies $\langle c^p \rangle$ is semiregular on each part and then $\langle c \rangle$ is semiregular on $V(\Gamma)$. Therefore, $\langle c \rangle \cap \langle a \rangle = 1$. Since $[c, b] = c^{-1}b^{-1}cb = c^{-1}ba^{\frac{p^e(p-1)}{2}} = c^{-2}$, we have $c \in \langle c^{-2} \rangle \in G'$. Then $G = \langle a, b \rangle = \langle a, c \rangle \leq G' \langle a \rangle$ and thus $G = \langle a, c \rangle = G' \langle a \rangle$. Since $c \in G'$ and G' is abelian, G' is semiregular on $V(\Gamma)$ and so $G' \cap \langle a \rangle = 1$. Therefore, $|G'| = |G|/|\langle a \rangle| = p^{e+1}$. On the other hands, by [3, Corollary C], we have $G' \cong G'/G' \cap \langle a \rangle$, which is isomorphic to a subgroup of $\langle g \rangle$. Thus, G' is a cyclic group of order p^{e+1} . Since $c \in G'$ and $\langle c \rangle$ permutes the parts, we have $G' = \langle c \rangle$ and hence $G = \langle c \rangle : \langle a \rangle$.

From the above fact, we can set $c^a = c^r$. Since $\langle a \rangle$ is core-free, $c^{a^i} \neq c$ for any $i \not\equiv 1 \pmod{p^{e+1}}$. Therefore, r is of order $\phi(p^{e+1})$ in $\mathbb{Z}_{\phi(p^{e+1})}$. Take two such r and r' and denote the corresponding groups by $G(r)$ and $G(r')$. Set $r = r'^s$ for some integer s . Then the map $\sigma : a \rightarrow a^s, c \rightarrow c$ gives an isomorphism from $G(r)$ to $G(r')$. Therefore, r can be chosen to be any given generator of $\mathbb{Z}_{p^{e+1}}^*$.

Now G satisfies all the relation of G_1 . A direct checking shows that $|G_1(p, e)| = |G|$ and so $G \cong G_1$, which is uniquely determined by p and e .

(2) Determination of isomorphic classes of maps.

By Lemma 2.3, for any $j \in \mathbb{Z}_{p^e(p-1)}^*$, $\mathcal{M}(G_1; a^j, b)$ is really a regular embedding of $K_{m[n]}$. Suppose that for two such parameters j_1 and j_2 , $\mathcal{M}(G_1; a^{j_1}, b) \cong \mathcal{M}(G_1; a^{j_2}, b)$. Then there exists a $\sigma \in \text{Aut}(G_1)$ such that $\sigma(a^{j_1}) = a^{j_2}$ and $\sigma(b) = b$. Then

$$\sigma(c) = \sigma(a^{\frac{p^e(p-1)}{2}}b) = \sigma(a^{j_1 \frac{p^e(p-1)}{2}}b) = a^{j_2 \frac{p^e(p-1)}{2}}b = a^{\frac{p^e(p-1)}{2}}b = c.$$

Therefore,

$$c^{r^{j_2}} = c^{a^{j_2}} = (\sigma(c))^{\sigma(a^{j_1})} = \sigma(c^{a^{j_1}}) = \sigma(c^{r^{j_1}}) = c^{r^{j_1}}.$$

It follows that $r^{j_1} \equiv r^{j_2} \pmod{p^{e+1}}$ and hence $j_1 \equiv j_2 \pmod{p^e(p-1)}$. Hence, get we $p^{e-1}(p-1)\phi(p-1)$ the maps in these family, while each such map is chiral.

Set $l = j - \frac{p^e(p-1)}{2}$. Then for any integer i , $(a^j b)^i = (a^l c)^i = a^{li} c^{\frac{r^{li}-1}{r^l-1}}$. If $p \equiv 1 \pmod{4}$, $|a^j b| = |r^l| = p^e(p-1)$; and if $p \equiv 1 \pmod{4}$, $|a^j b| = |r^l| = \frac{p^e(p-1)}{2}$. Therefore, the map has the type $\{p^e(p-1), p^e(p-1)\}$ if $p \equiv 1 \pmod{4}$ or $\{\frac{p^e(p-1)}{2}, p^e(p-1)\}$ if $p \equiv 3 \pmod{4}$. \square

Remark 3.2. In Lemma 3.1, $H = \langle x, z \rangle$, where $x = a^{p^{-1}}$ and $z = c^p$. Now $z^x = z^{r^{p^{-1}}}$, that is $q = r^{p^{-1}} = 1 + p^f$ under the notation in Eq (1). Suppose that $e \geq 2$. Then H is nonabelian. Since $|r| = \phi(p^e) = p^{e-1}(p-1)$, $p \equiv 1 + p \pmod{p^2}$, that is $p \parallel (r-1)$ and then $p \parallel (r^{p^{-1}} - 1)$, which implies $f = 1$.

3.2 $\text{Exp}(P) = p^e$ and H is nonabelian

Lemma 3.3. Suppose that $\text{Exp}(P) = p^e$ and H is nonabelian. Then $p = 3$, $e = 2$ and $\mathcal{M} \cong \mathcal{M}_5(1, l, j)$, where $l = 0, \pm 1$ and $j = \pm 1$. The number of the maps is 6. Moreover, each such map is chiral, with the type $\{3, 18\}$ if $l = 0$ and $\{9, 18\}$ if $l = \pm 1$.

The proof of Lemma 3.3 consists of the following Lemmas 3.4–3.7.

Let H , z , x and $q = 1 + p^f$ be as same as before. Since H is nonabelian, $e \geq 2$ and $1 \leq f \leq e - 1$.

Lemma 3.4. $N := \langle x^{p^{e-f}}, z \rangle \trianglelefteq G$.

Proof. We knew that $Z(H) = \langle x^{p^{e-f}}, z^{p^{e-f}} \rangle$. Let g be any element in G . Since $x^{p^{e-f}} \in Z(H) \trianglelefteq G$, we get $(x^{p^{e-f}})^g \in Z(H)^g \leq Z(H) \leq N$. Set $z^g = x^i z^j$. By Eq.(2), we have $(x^i z^j)^{p^f} = x^{ip^f} z^{j(q^{ip^f}-1)/(q^i-1)}$. On the other hand, since $H' = \langle z^{p^f} \rangle \trianglelefteq G$, we have $(x^i z^j)^{p^f} = (z^g)^{p^f} = (z^{p^f})^g \in \langle z^{p^f} \rangle$. Therefore $x^{ip^f} z^{j(q^{ip^f}-1)/(q^i-1)} \in \langle z^{p^f} \rangle$ and then $ip^f \equiv 0 \pmod{p^e}$. It follows that $p^{e-f} \mid i$ and so $z^g \in N$. Hence, $N \trianglelefteq G$. \square

Lemma 3.5. For any $g \in P \setminus H$, set $x^g = x^i z^j$ and $z^g = x^k z^l$. Then we have

- (1) $p^{e-f} \mid k$, $p^{e-f} \mid (i-1)$, $(j, p) = 1$ and $l \equiv 1 \pmod{p}$;
- (2) Rechoosing z , one may have $x^g = xz$.

Proof. (1) By the proof of Lemma 3.4, one gets $p^{e-f} \mid k$ and then $(l, p) = 1$. Then $x^{kq} = x^k$ and $x^k \in Z(H)$. Since

$$(z^g)^q = (x^k z^l)^q = x^{kq} z^{lq} = x^k z^{lq}, \quad (z^g)^g = (z^x)^g = (z^g)^{x^g} = (x^k z^l)^{x^i z^j} = x^k (z^l)^{x^i} = x^k z^{lq^i},$$

we get $x^k z^{lq} = x^k z^{lq^i}$ and then $lq \equiv lq^i \pmod{p^e}$. Since $q = 1 + p^f$ and $(l, p) = 1$, we have $p \nmid lq$ and hence $q^{i-1} \equiv 1 \pmod{p^e}$. Then $p^{e-f} \mid (i-1)$.

Write $\bar{G} = G/\Phi(H)$. Then $\bar{x}^{\bar{g}} = \bar{x}^{i-1} \bar{x} \bar{z}^j = \bar{x} \bar{z}^j$ and $\bar{z}^{\bar{g}} = \bar{x}^k \bar{z}^l = \bar{z}^l$. Noting $\langle x \rangle$ fixes γ_{11} and g permutes p parts, we get $\bar{x}^{\bar{g}} \neq \bar{x}$ and thus \bar{g} is faithfully represented on $\bar{H} \cong \mathbb{Z}_p^2$, acting as a matrix $\begin{pmatrix} 1 & 0 \\ j & l \end{pmatrix} \in \text{GL}(2, p)$. Since \bar{g} is of order p , we get $l \equiv 1 \pmod{p}$ and $(j, p) = 1$.

- (2) Since $x^g = x(x^{i-1} z^j)$ and

$$(x^{i-1} z^j)^x = x^{i-1} (z^j)^x = x^{i-1} z^{jq} = (x^{i-1} z^j)^q,$$

one may assume that $x^g = xz$, by replacing z by $x^{i-1} z^j$. \square

Lemma 3.6. *With the above notion, we have $p = 3$, $e = 2$ and $f = 1$.*

Proof. Recall $c = a^{\frac{p^e(p-1)}{2}}b$ and let $x^{c^k} = x^{u_k}z^{v_k}$ and $z^{c^k} = x^{s_k}z^{t_k}$ for all $k \geq 1$. By Lemma 3.5, we can set $u_1 = v_1 = 1$, $s_1 = s$ and $t_1 = t$ for some $s, t \in \mathbb{Z}_{p^e}$ such that $p^{e-f}|s$ and $t \equiv 1 \pmod{p}$. In particular, $x^s \in Z(H)$. Now we divide the proof into the following three steps.

Step 1. Show that

$$\begin{pmatrix} u_k & s_k \\ v_k & t_k \end{pmatrix} \equiv \begin{pmatrix} 1 & s \\ 1 & t \end{pmatrix}^k \pmod{p^e} \text{ for all } k \geq 1. \quad (3)$$

We are carrying out the proof by induction on k . The assertion is trivial if $k = 1$ and let $k \geq 2$. By Eq.(2), we have

$$(xz)^{u_{k-1}} = x^{u_{k-1}}z^{(q^{u_{k-1}}-1)/(q-1)}, (x^s z^t)^{v_{k-1}} = x^{sv_{k-1}}z^{tv_{k-1}}.$$

From Lemma 3.5.(1) and Lemma 2.2, we get

$$(q^{u_{k-1}} - 1)/(q - 1) \equiv u_{k-1} \pmod{p^e}, (q^{s_{k-1}} - 1)/(q - 1) \equiv s_{k-1} \pmod{p^e}.$$

Therefore,

$$\begin{aligned} x^{c^k} &= (x^{u_{k-1}}z^{v_{k-1}})^c = (x^c)^{u_{k-1}}(z^c)^{v_{k-1}} = (xz)^{u_{k-1}}(x^s z^t)^{v_{k-1}} \\ &= x^{u_{k-1}}z^{(q^{u_{k-1}}-1)/(q-1)}x^{sv_{k-1}}z^{tv_{k-1}} = x^{u_{k-1}+sv_{k-1}}z^{(q^{u_{k-1}}-1)/(q-1)+tv_{k-1}} \\ &= x^{u_{k-1}+sv_{k-1}}z^{u_{k-1}+tv_{k-1}}. \end{aligned}$$

Similarly, we get $z^{c^k} = x^{s_{k-1}+st_{k-1}}z^{s_{k-1}+tt_{k-1}}$. Therefore,

$$\begin{pmatrix} u_k & s_k \\ v_k & t_k \end{pmatrix} \equiv \begin{pmatrix} 1 & s \\ 1 & t \end{pmatrix} \begin{pmatrix} u_{k-1} & s_{k-1} \\ v_{k-1} & t_{k-1} \end{pmatrix} \pmod{p^e}.$$

Then we get the conclusion by employing the inductive hypothesis.

Step 2. Show that $p \parallel v_p$ if $p \geq 5$.

Suppose that $p \geq 5$. Set $A = \begin{pmatrix} 0 & s \\ 1 & t-1 \end{pmatrix}$. Then

$$A^2 \equiv 0 \pmod{p}, A^4 \equiv 0 \pmod{p^2}.$$

By Eq.(3), we have

$$\begin{pmatrix} u_p & s_p \\ v_p & t_p \end{pmatrix} = (E + A)^p \equiv E + pA = \begin{pmatrix} 1 & ps \\ p & 1+p(t-1) \end{pmatrix} \pmod{p^2},$$

which implies $p \parallel v_p$.

Step 3. Show $p = 3, e = 2$ and $f = 1$.

Set $c^p = x^i z^j$. Since $\text{Exp}(P) = p^e$, $p \mid i$ and $p \mid j$. Then

$$x^{c^p} = x^{x^i z^j} = x^{z^j} = x z^{-jq} z^j = x z^{-jp^f} \text{ and } z^{c^p} = z^{x^i z^j} = z^{x^i} = z^{q^i},$$

which implies $v_p \equiv -jp^f \pmod{p^e}$.

First suppose that $p \geq 5$. Since $p \parallel v_p$ by *Step 2*, we get $f = 1$ and $(p, j) = 1$. This forces $|c| = p^{e+1}$, contradicting to $\text{Exp}(P) = p^e$.

Next, suppose $p = 3$. Computing directly, one gets

$$\begin{pmatrix} 1 & s \\ 1 & t \end{pmatrix}^3 = \begin{pmatrix} 1+s(t+2) & s(1+t+t^2+s) \\ 1+t+t^2+s & s+2st+t^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -jp^f & q^i \end{pmatrix} \quad (4)$$

Solving Eq(4), we get $e = 2$ and $f = 1$. □

Lemma 3.7. *Suppose that $p = 3$, $f = 1$ and $e = 2$. Then $\mathcal{M} \cong \mathcal{M}_5(1, l, j)$, where $l = 0, \pm 1$ and $j = \pm 1$. The number of the maps is 6. Moreover, each such map is chiral, with the type $\{3, 18\}$ if $l = 0$ or $\{9, 18\}$ if $l = \pm 1$.*

Proof. Recall

$$G_5 = G_5(1, l) = \langle a, b \mid a^{18} = b^2 = 1, a^2 = x, x^b = y, [x, y] = x^3 y^{-3}, y^a = x^{-1} y^{-1}, (ab)^3 = x^{3l} y^{-3l} \rangle$$

(1) Show $G \cong G_5(1, l)$.

By the Eq.(8.1) in [15], we can set

$$H = \langle x, y \mid x^9 = y^9 = [x^3, y] = [x, y^3] = 1, [x, y] = x^3 y^{-3} \rangle.$$

Let $z = x^{-1}y$. Then one may rewrite $H = \langle x, z \rangle$ such that $z^x = z^4$. Write $\overline{G} = G/H$. Then $\overline{G} = \langle \overline{a}, \overline{b} \rangle \cong S_3$, where $|\overline{a}\overline{b}| = 3$.

Noting that $a^2 = x$, $b^2 = 1$, $x^b = y = xz$ and $z^b = y^{-1}x = z^{-1}$, G is determined by setting $z^a = x^u z^v$, $(ab)^3 = x^s z^t$ for some integers u, v, s and t . Lemma 3.4 tells us $3 \mid u$ but $3 \nmid v$. Since $\text{Exp}(P) = 9$, we have that 3 divides both s and t , and hence $(ab)^3 \in Z(H)$. Since

$$x = x^{(ab)^3} = x^{2u-uv+1} z^{-2uv+2u+v^2-v+1},$$

$$2u - uv + 1 \equiv 1 \pmod{9} \text{ and } -2uv + 2u + v^2 - v + 1 \equiv 0 \pmod{9}. \quad (5)$$

Since $z^4 = z^x = z^{a^2} = (x^u z^v)^a = x^u (x^u z^v)^v = x^{u+uv} z^{v^2}$, we have

$$u + uv \equiv 0 \pmod{9} \text{ and } v^2 \equiv 4 \pmod{9}. \quad (6)$$

By Eq. (5) and (6), we have $u \equiv -3 \pmod{9}$ and $v \equiv 2 \pmod{9}$. It follows that $z^a = x^{-3}z^2$ and hence $y^a = xx^{-3}z^2 = x^{-1}y^{-1}$. Since

$$x^s z^t = (x^s z^t)^{ab} = (x^s (x^{-3}z^2)^t)^b = (x^s z^{2t})^b = x^s z^{s+t},$$

we have $s \equiv 0 \pmod{9}$ and hence $(ab)^3 = z^t = x^{-t}y^t = x^{3l}y^{-3l}$ for $l = 0$ or ± 1 .

Now G satisfies all the relation of $G_5(1, l)$. A direct checking shows that $|G_5(1, l)| = |G|$ and so $G \cong G_5(1, l)$.

(2) Show $\mathcal{M} \cong \mathcal{M}_5(1, l, \pm 1)$.

It is easy to check that $\tau : a \rightarrow a^j, b \rightarrow b$ can be extended to an automorphism of G if and only if $j \equiv 1 \pmod{3}$. So \mathcal{M} is isomorphic to $\mathcal{M}(G; a^j, b)$ where $j = \pm 1$. Clearly, these maps are chiral. Since $|a^{\pm 1}b| = 3$ for $l = 0$ or 9 for $l = 1$, the maps have the type $\{3, 18\}$ if $l = 0$ or $\{9, 18\}$ if $l = \pm 1$. \square

3.3 $\text{Exp}(P) = p^e$ and H is abelian

Lemma 3.8. *Suppose that H is abelian. Then \mathcal{M} is isomorphic to*

- (1) $\mathcal{M}_2(p, j)$ where $j \in \mathbb{Z}_{p-1}^*$ for $p \geq 3$. The number of maps is $\phi(p-1)$. Moreover, each such map is chiral, with the type $\{p(p-1), p^e(p-1)\}$ if $p \equiv 1 \pmod{4}$ or $\{\frac{p(p-1)}{2}, p^e(p-1)\}$ if $p \equiv 3 \pmod{4}$.
- (2) $\mathcal{M}_3(1, e, l, j)$ where $e \geq 2$ and where $p = 3, (l, j) = (0, 1), (1, \pm 1)$. For $l = 0$ (resp $l = 1$), the number of maps is 1 (resp 2), the map is reflexible (resp chiral), with the type $\{3, 2 \cdot 3^e\}$ (resp $\{9, 2 \cdot 3^e\}$).

Proof. Suppose that H is abelian. If $\text{Exp}(P) = p^e = p$, then by [7] \mathcal{M} is isomorphic to $\mathcal{M}_2(p, j)$ where $j \in \mathbb{Z}_{p-1}^*$. If $\text{Exp}(P) = 3^e$ where $e \geq 3$, then by [24, Lemma 5.2], $\mathcal{M} \cong \mathcal{M}_3(1, e, l, j)$ where $(l, j) = (0, 1), (1, \pm 1)$. To prove this lemma, it suffices to show that $e = 1$ for $p \geq 5$.

Recall the notation, $\langle a \rangle = G_{\gamma_{11}}$ and b reverses the arc $(\gamma_{11}, \gamma_{21})$. Let $x = a^{m-1}$ and $y = x^b$. Since H is abelian, $H = \langle x, y \rangle$. Set $c = a^{\frac{p^e(p-1)}{2}}b$. Now $\overline{G} = G/H = \langle \overline{a}, \overline{c} \rangle \cong \text{AGL}(1, p)$. Since the product of two different involutions in $\text{AGL}(1, p)$ must be of order p , $|\overline{c}| = p$ and so c is a p -element. Considering the conjugacy actions of a and c on H , we may set

$$x^a = a, y^a = x^s y^t, x^c = y, y^c = x^w y^r. \quad (7)$$

Now we are going to determine s, t, w and r .

For any $1 \leq i \leq p-2$, since a^i moves the block Δ_2 to another block and $y = (a^{p-1})^b$ fixes a point $\gamma_{22} \in \Delta_2$ and acts no fixed point in Δ_j where $j \neq 2$, we know that $\langle y^{a^i} \rangle$

fixes a point $\gamma_{22}^{a^i}$ and acts no fixed point in Δ_2 , which implies $\langle y^{a^i} \rangle \cap \langle y \rangle = 1$. From $y^{a^i} = x^{s(1+t+\dots+t^{i-1})} y^{t^i}$, we get $t^i \neq 1$ in $Z_{p^e}^*$. In particular, $t \neq \pm 1 \pmod{p}$. Since

$$y^c = \left(x^{s(1+t+\dots+t^{\frac{p^e(p-1)}{2}-1})} y^{t^{\frac{p^e(p-1)}{2}}} \right)^b = (x^{\frac{2s}{1-t}} y^{-1})^b = x^{-1} y^{\frac{2s}{1-t}},$$

we get $w = -1$ and $r = \frac{2s}{1-t}$.

The conjugacy action of G on H gives a unfaithful homomorphism π from G to $\text{Aut}(H)$. By Eq(7), $\pi(a)$ and $\pi(c)$ are represented as two matrixes

$$A = \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & -1 \\ 1 & r \end{pmatrix}$$

in $\text{GL}(2, \mathbb{Z}_{p^e})$. Set $C^k = \begin{pmatrix} \alpha_k & \mu_k \\ \beta_k & \lambda_k \end{pmatrix}$ for all $k \geq 1$. Then by computing $CC^k = C^kC$, one may get

$$\begin{pmatrix} -\beta_k & -\lambda_k \\ \alpha_k + r\beta_k & \mu_k + r\lambda_k \end{pmatrix} = \begin{pmatrix} \mu_k & -\alpha_k + r\mu_k \\ \lambda_k & -\beta_k + r\lambda_k \end{pmatrix},$$

which implies

$$\mu_k \equiv -\beta_k \pmod{p^e}. \quad (8)$$

Since $\langle \bar{c} \rangle \triangleleft \bar{G}$, we set $a^{-1}ca = c^k z$, for some $z \in H$. Form this, we get $A^{-1}CA = C^k$. Since

$$A^{-1}CA = \begin{pmatrix} -st^{-1} & -s^2t^{-1} - t - rs \\ t^{-1} & st^{-1} + r \end{pmatrix},$$

by Eq(8) we get

$$t^{-1} \equiv s^2t^{-1} + t + rs \pmod{p^e}.$$

Inserting $r = \frac{2s}{1-t}$, we obtain

$$(1+t)(s-t+1)(s+t-1) \equiv 0 \pmod{p^e}. \quad (9)$$

Considering the induced action of c in $H/\Phi(H)$ where $\Phi(H) = \langle x^p, y^p \rangle$, we get $C^p = E \pmod{p}$. Noting that any matrix of order p in $\text{GL}(2, p)$ has the eigenvalue 1, one gets $r \equiv 2 \pmod{p}$, that is $s+t-1 \equiv 0 \pmod{p}$. Noting $t \not\equiv \pm 1 \pmod{p}$, we get $s-t+1 = (s+t-1) - 2(t-1) \not\equiv 0 \pmod{p}$. Thus we get from Eq(9), that $s+t-1 \equiv 0 \pmod{p^e}$, which follows that $r = \frac{2s}{1-t} \equiv 2 \pmod{p^e}$. Therefore,

$$C^p = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^p = \begin{pmatrix} -p+1 & -p \\ p & p+1 \end{pmatrix} = E,$$

that is $e = 1$. □

4 Regular embeddings of $K_{3[n]}$

In this section, we assume that $m = 3$ and $n = 3^e k$ where $e \geq 0$, $3 \nmid k$ and $k \geq 2$. Again, let $\Gamma = K_{3[n]}$, $V(\Gamma)$, $E(\Gamma)$, \mathcal{M} , $G = \text{Aut}(\mathcal{M}) = \langle a, b \rangle$, $H = \text{Aut}_0(\mathcal{M})$ be as in Section 4. Then by Proposition 1.1, $H = Q \times K$ where Q is a 3-group and K is an abelian 3'-group. Let P be a Sylow 3-subgroup of G . Naturally, we divide the discussions into three cases: Q is abelian, Q is nonabelian and $\text{Exp}(P) = 3^{e+1}$, and Q is nonabelian and $\text{Exp}(P) = 3^e$, separately.

Proposition 4.1. *[24, Lemma 5.2] Suppose that Q is abelian and $k \geq 2$. Then \mathcal{M} is isomorphic to one of the maps $\mathcal{M}_3(k, e, l, j)$, where $(l, j) = (0, 1)$ for $e = 0$ and $(l, j) = (0, 1), (1, 1)$ or $(1, -1)$ for $e \geq 1$. For $l = 0$ (resp $l = 1$), the number of maps is 1 (resp 2), the maps are reflexible (resp chiral), with the type $\{3, 2 \cdot 3^e\}$ (resp $\{9, 2 \cdot 3^e\}$).*

Lemma 4.2. *If Q is nonabelian and $\text{Exp}(P) = 3^{e+1}$ where $e \geq 2$. Then $\mathcal{M} \cong \mathcal{M}_4(k, e, j)$ where $k \geq 2$, $j \in \mathbb{Z}_{2k \cdot 3^e}^*$ and $\mathcal{M}_4(k, e, j_1) \cong \mathcal{M}_4(k, e, j_2)$ if and only if $j_1 \equiv j_2 \pmod{2 \cdot 3^e}$. The number of maps is $2 \cdot 3^{e-1}$. Moreover, the maps are chiral with the type $\{3^{e+1}, 2 \cdot 3^e k\}$.*

Proof. Recall that

$$G_4 = G_4(k, e) = \langle a, b \mid a^{2 \cdot 3^e k} = b^2 = 1, c = a^{3^e} b, a^{2 \cdot 3^e} = x_1, x_1^b = y_1, [x_1, y_1] = 1, \\ y_1^a = x_1^{-1} y_1^{-1}, c^{3^{e+1}} = 1, c^a = c^2 x_1^u y_1^{\frac{u-1}{2}} \rangle,$$

where $3 \nmid k$, $k \geq 2$, $e \geq 2$ and $u3^e \equiv 1 \pmod{k}$.

(1) Show $G \cong G_4$.

Set $x = a^2$, $y = x^b$, $x_1 = a^{2 \cdot 3^e}$, $y_1 = x_1^b$ and $c = a^{3^e} b$. Then $K = \langle x_1 \rangle \times \langle y_1 \rangle$. We are going to determine group G by considering two groups $\langle K, c \rangle$ and $\langle K, c \rangle \langle a \rangle$, separately. Before doing that, please note the following two remarks about groups G/Q and G/K .

Set $\tilde{G} = G/Q$ and let $\tilde{\mathcal{M}}$ to denote the quotient map of \mathcal{M} induced by Q . Then $\tilde{H} = HQ/Q$ is abelian, $\tilde{\mathcal{M}}$ is a map of family of $\mathcal{M}_3(k, 0, 0, \pm 1)$ and $\text{Aut}(\tilde{\mathcal{M}}) = \tilde{G}$ has the following presentation:

$$\tilde{G} = \langle \tilde{a}, \tilde{b} \mid \tilde{a}^{2k} = \tilde{b}^2 = (\tilde{a}\tilde{b})^3 = \tilde{1}, \tilde{a}^2 = \tilde{x}, \tilde{x}^{\tilde{b}} = \tilde{y}, [\tilde{x}, \tilde{y}] = \tilde{1}, \tilde{y}^{\tilde{a}} = \tilde{x}^{-1} \tilde{y}^{-1} \rangle. \quad (10)$$

Then, we get $\tilde{y}_1^{\tilde{a}} = (\tilde{y}^{3^e})^{\tilde{a}} = \tilde{x}^{-3^e} \tilde{y}^{-3^e} = \tilde{x}_1^{-1} \tilde{y}_1^{-1}$ and then $y_1^a y_1 x_1 \in Q \cap K = 1$, that is $y_1^a = x_1^{-1} y_1^{-1}$.

Set $\overline{G} = G/K$ and let $\overline{\mathcal{M}}$ to denote the quotient map of \mathcal{M} induced by K . Then $\text{Aut}(\overline{\mathcal{M}}) = \overline{G}$. By the proof of Lemma 3.1, we know that in \overline{G} ,

$$\overline{G}' = \langle \overline{c} \rangle, \quad \overline{c}^{\overline{a}} = \overline{c}^r. \quad (11)$$

It is easy to see 2 is of order $2 \cdot 3^e$ in $Z_{3^{e+1}}^*$ and so we let $r = 2$.

Now we show two facts:

Fact 1: $\langle K, c \rangle = K : \langle c \rangle \triangleleft G$, where $|c| = 3^{e+1}$, $x_1^c = y_1$, $y_1^c = x_1^{-1}y_1^{-1}$, $[c^3, x_1] = [c^3, y_1] = 1$.

Since $\langle \bar{c} \rangle \triangleleft \bar{G}$, $\langle K, c \rangle \triangleleft G$. Write $g = ab$. By Eq(10), we get $\tilde{g}^3 = 1$. Since $\langle \tilde{a}^2 \rangle = \langle \tilde{x} \rangle = \langle \tilde{x}_1 \rangle$, we get $\tilde{a}^{3^e-1} \in \langle \tilde{x}_1 \rangle$. Let $\tilde{a}^{3^e-1} = \tilde{x}_1^i$ for some integer i . Then $\tilde{c} = \tilde{a}^{3^e} \tilde{b} = \tilde{a}^{3^e-1} \tilde{a} \tilde{b} = \tilde{x}_1^i \tilde{g}$. Since $\tilde{x}_1^{\tilde{g}} = \tilde{x}_1^{\tilde{a} \tilde{b}} = \tilde{x}_1^{\tilde{b}} = \tilde{y}_1$ and $\tilde{y}_1^{\tilde{g}} = \tilde{y}_1^{\tilde{a} \tilde{b}} = (\tilde{x}_1^{-1} \tilde{y}_1^{-1})^{\tilde{b}} = \tilde{x}_1^{-1} \tilde{y}_1^{-1}$, we have

$$(\tilde{c})^3 = \tilde{g}^3 (\tilde{x}_1^i)^{\tilde{g}^3} (\tilde{x}_1^i)^{\tilde{g}^2} (\tilde{x}_1^i)^{\tilde{g}} = \tilde{x}_1^i \tilde{x}_1^{-i} \tilde{y}_1^{-i} \tilde{y}_1^i = 1.$$

which implies $c^3 \in Q$. Noting $\text{Exp}(Q) = 3^e$, we have $|c| \leq 3^{e+1}$. Since the order of \bar{c} is 3^{e+1} in \bar{G} , $|c| \geq 3^{e+1}$. Therefore, $|c| = 3^{e+1}$.

Since $y_1^{a^i} = x_1^{-1}y_1^{-1}$ for any odd i , and y_1 for any even i , we get

$$x_1^c = x^{a^{3^e}b} = x_1^b = y_1, y_1^c = x_1^{a^{3^e}b} = (x_1^{-1}y_1^{-1})^b = x_1^{-1}y_1^{-1}, [c^3, x_1] = [c^3, y_1] = 1.$$

Fact 2: $G = \langle K, c \rangle \langle a \rangle$, where $c^a = c^2 x^u y^{\frac{u-1}{2}}$ where $u3^e \equiv 1 \pmod{k}$.

By (11), let $c^a = c^2 x_1^u y_1^v$ for some integers u and v . First, one can deduce the following formula by induction :

$$(c^2 x_1^u y_1^v)^i = \begin{cases} c^{2i}, & i \equiv 0 \pmod{3} \\ c^{2i} x_1^u y_1^v, & i \equiv 1 \pmod{3} \\ c^{2i} x_1^v y_1^{v-u}, & i \equiv 2 \pmod{3} \end{cases}, \quad c^{a^{2i}} = c^{4^i} x_1^{iu} y_1^{-iu}. \quad (12)$$

Then,

$$c^{a^{2 \cdot 3^e}} = c^{4^{3^e}} x_1^{3^e u} y_1^{-3^e u} = c x_1^{3^e u} y_1^{-3^e u}, \quad c^{x_1} = c(x_1^{-1})^c x_1 = c x_1 y_1^{-1}.$$

Since $a^{2 \cdot 3^e} = x_1$, we get $u3^e \equiv 1 \pmod{k}$.

To show $v \equiv \frac{u-1}{2} \pmod{k}$, note that $2^{3^e} \equiv 2 \pmod{3}$ and $2^{3^e-1} \equiv 1 \pmod{3}$. Set $2^{3^e} = 3w + 2$. By Fact 1 and Eq(12), we get

$$c^{a^{3^e}} = (c^{2^{3^e-1}} x_1^{\frac{3^e-1}{2}u} y_1^{\frac{1-3^e}{2}u})^a = (c^2 x_1^u y_1^v)^{2^{3^e-1}} x_1^{(3^e-1)u} y_1^{\frac{3^e-1}{2}u} = c^{2^{3^e}} x_1 y_1^{\frac{u-1}{2}+v} \quad (13)$$

and then

$$\begin{aligned} (c^{a^{3^e}})^{2^{3^e}} &= (c^{2^{3^e}} x_1 y_1^{\frac{u-1}{2}+v})^{2^{3^e}} = (c^{3w} c^2 x_1 y_1^{\frac{u-1}{2}+v})^{2^{3^e}} \\ &= c^{3w \cdot 2^{3^e}} c^{2 \cdot 2^{3^e}} x_1^{\frac{1-u}{2}+v} y_1^{-\frac{u+1}{2}+v} = c^{2^{2 \cdot 3^e}} x_1^{\frac{1-u}{2}+v} y_1^{-\frac{u+1}{2}+v}. \end{aligned} \quad (14)$$

Since $c = a^{3^e}b$ and $cb = a^{3^e}$, we have $c^b = c^{a^{3^e}}$. Then by (13) and (14), we get

$$\begin{aligned} c &= c^c = c^{a^{3^e}b} = (c^{2^{3^e}} x_1 y_1^{\frac{1-u}{2}+v})^b = (c^{a^{3^e}})^{2^{3^e}} x_1^{\frac{1-u}{2}+v} y_1 \\ &= c^{2^{2 \cdot 3^e}} x_1^{\frac{1-u}{2}+v} y_1^{-\frac{u+1}{2}+v} x_1^{\frac{1-u}{2}+v} y_1 = c x_1^{1-u+2v} y_1^{\frac{1-u}{2}+v}. \end{aligned}$$

Therefore $v \equiv \frac{u-1}{2} \pmod{k}$.

Come back to the proof. From the above arguments, G satisfies all the relations of G_4 and then G is an homomorphic image of G_4 . By Fact 1, $\langle K, c \rangle$ is a split cyclic extension of K by $\langle c \rangle$ and $|\langle K, c \rangle| = k^2 3^{e+1}$. By Fact 2, one may check that $\langle K, c \rangle \langle a \rangle$ is a cyclic extension of $\langle K, c \rangle$ by $\langle a \rangle$ and then $|G_4| = |\langle K, c \rangle \langle a \rangle| = k^2 3^{e+1} \cdot (2 \cdot 3^e) = |G|$. Hence, $G_4 \cong G$. From the proof, one may see that G_4 is uniquely determined by k and e .

(2) Determination of isomorphic classes of maps

Suppose that $\mathcal{M}(G_4; a^{j_1}, b) \cong \mathcal{M}(G_4; a^{j_2}, b)$ for two parameters $j_1, j_2 \in \mathbb{Z}_{2k \cdot 3^e}^*$. Then there exists a $\sigma \in \text{Aut}(G_4)$ such that $\sigma(a^{j_1}) = a^{j_2}$ and $\sigma(b) = b$. Clearly, σ induces an automorphism $\bar{\sigma}$ of \bar{G} such that $\bar{\sigma}(\bar{a}^{j_1}) = \bar{a}^{j_2}$ and $\bar{\sigma}(\bar{b}) = \bar{b}$. By Lemma 3.1, we get $j_1 \equiv j_2 \pmod{2 \cdot 3^e}$.

Conversely, suppose $j_1 \equiv j_2 \pmod{2 \cdot 3^e}$ for $j_1, j_2 \in \mathbb{Z}_{2k \cdot 3^e}^*$. Then $j = j_2 j_1^{-1} \equiv 1 \pmod{2 \cdot 3^e}$. It suffices to show that there exists an isomorphism of G_4 mapping a to a^j and b to b , for any $j \in \mathbb{Z}_{2k \cdot 3^e}^*$ and $j \equiv 1 \pmod{2 \cdot 3^e}$. Let ϕ be a such isomorphism, that is $\phi(a) = a^j$ and $\phi(b) = b$. Then

$$\phi(x_1) = x_1^j, \phi(y_1) = y_1^j, \phi(c) = a^{j3^e}b = a^{\frac{j-1}{2} \cdot 2 \cdot 3^e} a^{3^e}b = x_1^{\frac{j-1}{2}} c.$$

By Fact 1 of (1), we have

$$\begin{aligned} \phi(c)^{3^{e+1}} &= (x_1^{\frac{j-1}{2}} c)^{3^{e+1}} = (c^3 x_1^{c^3} x_1^{c^2} x_1^c)^{3^e} = c^{3^{e+1}} = 1, \\ \phi(y_1)^{\phi(a)} &= y_1^{j a^j} = x_1^{-j} y_1^{-j} = \phi(x_1)^{-1} \phi(y_1)^{-1}. \end{aligned}$$

Set $j = 1 + 2i \cdot 3^e$. Then

$$\begin{aligned} \phi(c)^{\phi(a)} &= (x_1^{\frac{j-1}{2}} c)^{a^j} = x_1^{\frac{j-1}{2}} c^{a^j} = x_1^{\frac{j-1}{2}} c^{x_1^a} = x_1^{\frac{j-1}{2}} x_1^{-i} c^a x_1^i \\ &= x_1^{\frac{j-1}{2}-i} c^2 x_1^u y_1^v x_1^i = c^2 (x_1^{-1} y_1^{-1})^{\frac{j-1}{2}-i} x_1^u y_1^v x_1^i = c^2 x_1^{\frac{1-j}{2}+2i+u} y_1^{i+\frac{1-j}{2}+v}, \\ \phi(c)^2 \phi(x_1)^u \phi(y_1)^v &= (x_1^{\frac{j-1}{2}} c)^2 (x_1^j)^u (y_1^j)^v = c^2 (x_1^{-1} y_1^{-1})^{\frac{j-1}{2}} y_1^{\frac{j-1}{2}} x_1^{ju} y_1^{jv} = c^2 x_1^{\frac{1-j}{2}+ju} y_1^{jv}. \end{aligned}$$

Noting $3 \nmid k$, $j = 1 + 2i3^e$, $u3^e \equiv (\text{mod } k)$ and $v = \frac{u-1}{2}$, we get

$$2i + u \equiv ju \pmod{k} \Leftrightarrow (2i + u)3^e \equiv ju3^e \pmod{k} \Leftrightarrow j - 1 + 1 \equiv j \pmod{k},$$

$$\begin{aligned} i + \frac{1-j}{2} + v &\equiv jv \pmod{k} \Leftrightarrow i + \frac{1-j}{2} + (1-j)\frac{1-3^e}{2}u \equiv 0 \pmod{k} \\ \Leftrightarrow i3^e + \frac{1-j}{2}3^e + (1-j)\frac{1-3^e}{2} &\equiv 0 \pmod{k} \\ \Leftrightarrow \frac{j-1}{2} + \frac{1-j}{2}3^e + (1-j)\frac{1-3^e}{2} &\equiv 0 \pmod{k}. \end{aligned}$$

Hence, $\phi(c)^{\phi(a)} = \phi(c)^2 \phi(x_1)^u \phi(y_1)^v$ and thus ϕ is an homomorphism. Clearly, ϕ is a bijective and then $\phi \in \text{Aut}(G_4)$.

(3) Determination of the number and type of maps.

By (2), $\mathcal{M}(G_4; a^{j_1}, b) \cong \mathcal{M}(G_4; a^{j_2}, b)$ if and only if $j_1 \equiv j_2 \pmod{2 \cdot 3^e}$. Noting that for any $1 \leq i \leq 2 \cdot 3^e - 1$ and $(i, 2 \cdot 3^e) = 1$, the set $\{i + h \cdot 2 \cdot 3^e \mid 0 \leq h \leq k - 1\}$ contains at least one number which is coprime to $2k \cdot 3^e$, we have $\phi(2 \cdot 3^e) = 2 \cdot 3^{e-1}$ nonisomorphic maps in this family. With the same arguments as in Fact 1 of (1), one may check $|a^j b| = 3^{e+1}$. Thus, the map has the type $\{3^{e+1}, 2 \cdot 3^e k\}$. Clearly, the maps are chiral. \square

Lemma 4.3. *If Q is nonabelian and $\text{Exp}(P) = 3^e$, then $e = 2$, $\mathcal{M} \cong \mathcal{M}_5(k, l, j)$, where $l = 0, \pm 1$ and $j = \pm 1$. The number of maps is 6, the maps are chiral with the type $\{3, 18k\}$ for $l = 0$ and $\{9, 18k\}$ for $l = \pm 1$.*

Proof. Recall that

$$G_5 = G_5(k, l) = \langle a, b \mid a^{18k} = b^2 = 1, a^2 = x, x^b = y, [x, y] = x^{3k} y^{-3k}, y^c = x^{-1} y^{-1} (ab)^3 = x^{3lk} y^{-3lk} \rangle.$$

First, we determine the group G . Let $\overline{G} = G/K$. Since Q is nonabelian and $\text{Exp}(P) = 3^e$, we have $\overline{H} \cong \overline{Q}$ is nonabelian and $\text{Exp}(\overline{P}) = 3^e$. By Lemma 3.3, we have $e = 2$.

By the presentation of \tilde{G} in Eq. (10), we have $[\tilde{x}, \tilde{y}] = \tilde{1}$, $\tilde{y}^{\tilde{a}} = \tilde{x}^{-1} \tilde{y}^{-1}$ and $(\tilde{a} \tilde{b})^3 = \tilde{1}$. Therefore, we can set

$$[x, y] = x^{3ik} y^{j3k}, y^a = x^{-1} y^{-1} x^{uk} y^{vk} \text{ and } (ab)^3 = x^{3sk} y^{3tk},$$

and hence

$$[\overline{x}, \overline{y}] = \overline{x}^{3ik} \overline{y}^{3jk}, \overline{y}^{\overline{a}} = \overline{x}^{-1} \overline{y}^{-1} \overline{x}^{uk} \overline{y}^{vk} \text{ and } \overline{ab}^3 = \overline{x}^{3sk} \overline{y}^{3tk}. \quad (15)$$

Since $\langle \overline{x}, \overline{y} \rangle$ is a 3^2 -isobicycle group, no loss, we let $i = 1$ and $j = -1$. In viewing of the proof of Lemma 3.7, it has to be

$$u \equiv v \equiv 0 \pmod{9}, s \equiv -t \equiv l \pmod{9},$$

where $l = 0, \pm 1$. Thus G satisfies all the relations of G_5 and then G is an homomorphic image of G_5 . Since $|G_5| = |G|$, $G \cong G_5$. Clearly, the group G_5 is uniquely determined by k and e .

Similar to last lemma, we can get the maps $\mathcal{M}_5(k, l, j)$, where $j \neq \pm 1$. The other statements of the lemma are clear. \square

5 Proof of Theorem 1.2

Proof of Theorem 1.2: Let \mathcal{M} be a regular embedding of $\Gamma = K_{m[n]}$ where $m \geq 3$ and $n \geq 2$, and let $\text{Aut}_0(\mathcal{M})$ be the subgroup of $\text{Aut}(\mathcal{M})$ fixing each part setwise. By Proposition 1.1, either $\Gamma = K_{p[p^e]}$ where $p \geq 5$ is prime, or $\Gamma = K_{3[3^e k]}$ for $3 \nmid k$.

Suppose that $\Gamma = K_{p[p^e]}$ where $p \geq 3$. If $\text{Exp}(P) = p^{e+1}$, then $\mathcal{M} \cong \mathcal{M}_1(p, e, j)$, see Lemma 3.1; if $\text{Exp}(P) = p^e$ and H is nonabelian, then $\mathcal{M} \cong \mathcal{M}_5(1, l, j)$, see Lemma 3.3; and if $\text{Exp}(P) = p^e$ and H is abelian, then $\mathcal{M} \cong \mathcal{M}_2(p, j)$ or $\mathcal{M}_3(1, e, l, j)$, see Lemma 3.8.

Suppose that $\Gamma = K_{3[3^e k]}$ where $3 \nmid k$ and $k \geq 2$. If Q is abelian, then $\mathcal{M} \cong \mathcal{M}_3(k, e, l, j)$ where $k \geq 2$, see Proposition 4.1; if Q is nonabelian and $\text{Exp}(P) = 3^{e+1}$, then $\mathcal{M} \cong \mathcal{M}_4(k, e, j)$, see Lemma 4.2; and if Q is nonabelian and $\text{Exp}(P) = 3^e$, then $\mathcal{M} \cong \mathcal{M}_5(k, l, j)$ where $k \geq 2$, see Lemma 4.3.

For the proof, we may know that all these five families of maps are pairwise nonisomorphic. Moreover, all the dates in Table 1 and 2 can be extracted from the corresponding lemmas and propositions. \square

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